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Diffraction of Electromagnetic Waves by a Slit in a Conducting Plane Between Different Media

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DIFFRACTION OF ELECTROMAGNETIC WAVES BY A SLIT IN A CONDUCTING PLANE BETWEEN DIFFERENT MEDIA

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Abstract

The problem of diffraction of electromagnetic waves by an infinite slit in a perfect conductor at the interface between two different dielectric media is solved by the use of expansions of the fields in each medium in terms of appropriate Mathieu functions. By matching these expansions across the slit an infinite set of linear equations is obtained for the coefficients in the expansion of the diffracted fields. The solution is put in such a form that numerical results can be readily obtained.

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1. Introduction

One of the classical diffraction problems is the problem of diffraction of electromagnetic waves by an infinitely long parallel slit in a perfectly conducting plane embedded in a homogeneous medium. The problem was first treated by Sieger[1], and extensive numerical results are due especially to Morse and Rubenstein[2] and Skavlem[3]. The general procedure is to expand in terms of Mathieu functions the incident, reflected, and diffracted fields, and then to evaluate explicitly the coefficients of the series using the boundary conditions.

In the present paper we shall consider a modification of this problem: we assume that the medium on one side of the conducting plane differs in dielectric constant and magnetic permeability from the medium on the other side. The general procedure of expanding into a series of Mathieu functions is also applicable in this case, but the coefficients in the expansion of the diffracted field can no longer be evaluated explicitly; they must now be determined numerically for each wavelength by solving an infinite set of linear equations. Fortunately this infinite set reduces, in practice, to a finite set. (The number of equations increases with decreasing wavelength.) For wavelengths not smaller than one-third of the slit width, it does not seem too difficult to work out the numerical solution.

2. Formulation of the problem

Let $z > 0$ be a half-space with dielectric constant ε_1 and magnetic permeability μ_1, and let $z < 0$ be a half-space with corresponding quantities ε_2 and μ_2. At the interface $z = 0$ there is a perfectly conducting plane with a slit. We introduce an elliptic cylinder coordinate system by the transformation

$$x = a \cosh \xi \cos \eta , \quad y = y, \quad z = a \sinh \xi \sin \eta , \quad (0 \leq \xi < \infty, -\pi \leq \eta \leq \pi). \tag{1}$$

The slit, which is located at $-a \leq x \leq a$, is then a singularity of the coordinate systems, because the value of η assigned to each point on the slit may have two values, $\pm \eta$, depending on whether one approaches the slit from the upper or the lower half-space.

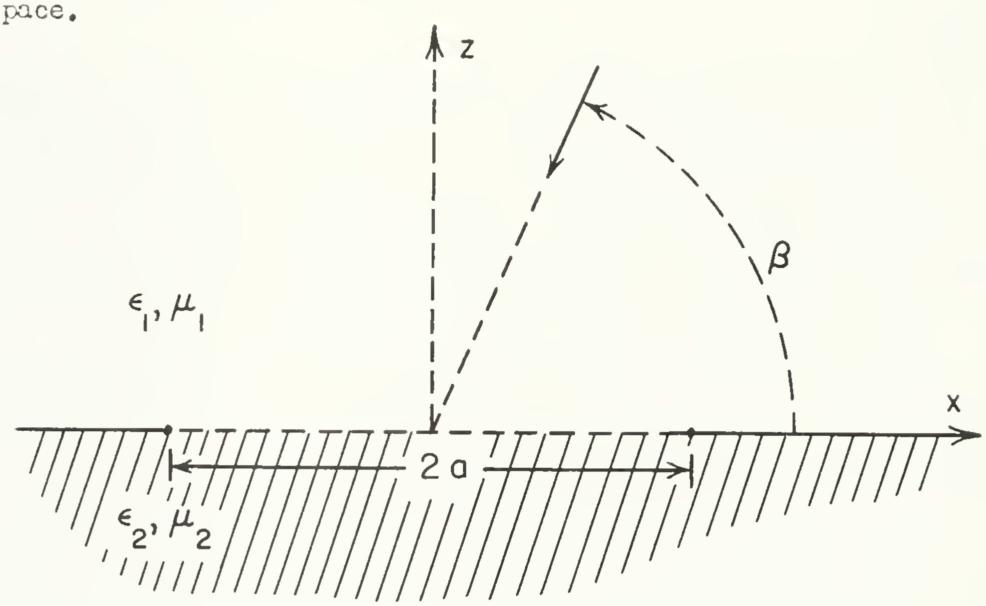

Figure 1

We restrict our discussion to time-harmonic fields with time dependence $e^{i\omega t}$, where ω is the circular frequency; in general, we shall omit this factor from our formulas. Maxwell's equations then read

$$i\omega\varepsilon \vec{E} = \text{curl } \vec{H}, \qquad - i\omega\mu \vec{H} = \text{curl } \vec{E} . \tag{2}$$

The boundary conditions are that

$$E_x = E_y = 0 \quad \text{for } z = 0, \ |x| \geq a, \ \text{i.e., for } \eta = 0 \text{ and } \eta = \pm \pi , \tag{3}$$

and that

$$E_x, E_y, H_x, H_y \text{ are continuous across the slit } z = 0, \ |x| \leq a, \text{ i.e., } \xi = 0 . \tag{4}$$

3. Solution for electric vector parallel to the slit

Let a plane wave be incident from the direction designated in Figure 1 by the angle β. Let the electric vector of this plane wave have only one component, E_i, which is parallel to the slit; if we denote the amplitude of E_i by E_o then

$$E_i = E_o \exp\left[ik_1(x\cos\beta + z\sin\beta)\right], \qquad z \geq 0 . \tag{5}$$

If the slit were absent, we would have a reflected wave with an electric vector parallel to the incident wave:

$$E_r = -E_o \exp\left[ik_1(x\cos\beta - z\sin\beta)\right], \qquad z \geq 0 . \tag{6}$$

We now expand each of these waves into a series of Mathieu functions;* we have, for $0 \leq \eta \leq \pi$,

$$\begin{aligned}\frac{E_i}{E_o} &= 2\sum_{m=0}^{\infty} i^m ce_m(\beta;\, h_1^2)\, Mc_m^{(1)}(\xi;\, h_1)\, ce_m(\eta;\, h_1^2)\\ &+ 2\sum_{m=1}^{\infty} i^m se_m(\beta;\, h_1^2)\, Ms_m^{(1)}(\xi;\, h_1)\, se_m(\eta;\, h_1^2),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\frac{E_r}{E_o} &= -2\sum_{m=0}^{\infty} i^m ce_m(\beta;\, h_1^2)\, Mc_m^{(1)}(\xi;\, h_1)\, ce_m(\eta;\, h_1^2)\\ &+ 2\sum_{m=1}^{\infty} i^m se_m(\beta;\, h_1^2)\, Ms_m^{(1)}(\xi;\, h_1)\, se_m(\eta;\, h_1^2),\end{aligned} \tag{8}$$

where

$$2h_1 = k_1 a, \quad k_1 = \omega\sqrt{\varepsilon_1 \mu_1} . \tag{9}$$

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* For the definitions and notations for the Mathieu functions used in this paper, see Meixner-Schaefke[4]. For the expansion of a plane wave see Section 2.86 of this book.

To meet all the conditions of the problem it will be seen that it is sufficient to assume that the electric vector of the diffracted field also has only a single component, E_d, which is parallel to the slit. Now we expand also this diffracted field E_d into a series of Mathieu functions:

$$(10) \qquad \frac{E_d}{E_o} = \sum_{m=1}^{\infty} V_{1m}\, Ms_m^{(4)}(\xi;\, h_1)se_m(\eta\, ;\, h_1^2) \qquad z \geq 0, \text{ i.e., } 0 \leq \eta \leq \pi\, ,$$

$$(11) \qquad \frac{E_d}{E_o} = \sum_{m=1}^{\infty} V_{2m}\, Ms_m^{(4)}(\xi;\, h_2)se_m(\eta\, ;\, h_2^2) \qquad z \leq 0, \text{ i.e., } -\pi \leq \eta \leq 0\, .$$

The use of the Mathieu functions of the fourth kind arises from the radiation condition, namely that the diffracted wave must be an outgoing wave. The omission of terms with Mathieu functions $ce_m(\eta\, ;\, h_1^2)$ and $ce_m(\eta\, ;\, h_2^2)$ in (10) and (11) respectively is again justified by the fact that these terms are not necessary for satisfying the requirements for the solution of our diffraction problem. The parameter h_2 is given by

$$(12) \qquad 2h_2 = k_2 a\, , \qquad k_2 = \omega\sqrt{\varepsilon_2\, \mu_2}\, .$$

The coefficient of the expansions, V_{im} $(i = 1,2)$, are to be determined.

On the perfectly conducting screen ($\eta = 0$, $\eta = \pm\pi$), $E_i + E_r$ vanishes; therefore E_d must also vanish there. This is indeed the case in equations (10) and (11), since the functions se_m vanish for $\eta = 0$ and $\eta = \pm\pi$.

In the slit $\xi = 0$ the conditions (4) must be satisfied; that is, for each value of η we must have

$$(4a) \qquad \begin{aligned} E_i + E_r + E_d\Big|_{\xi = 0,\, \eta \geq 0} &= E_d\Big|_{\xi = 0,\, \eta \leq 0}\, , \\ \frac{1}{\mu_1}\frac{\partial}{\partial z}(E_i + E_r + E_d)\Big|_{\xi = 0,\, \eta \geq 0} &= \frac{1}{\mu_2}\frac{\partial}{\partial z}E_d\Big|_{\xi = 0,\, \eta \leq 0}\, . \end{aligned}$$

(In the above equations, the positive value of η is used on the left-hand side and the same value of η with a negative sign is used on the right-hand side.)

We note that

$$\frac{\partial}{\partial z} = \frac{1}{a \sin\eta} \frac{\partial}{\partial \xi} \text{ for } \xi = 0; \qquad se_m(-\eta) = -se_m(\eta) .$$

Then, inserting the series expansions (7), (8), (10), (11) into (4a), we get, for $0 \le \eta \le \pi$,

$$\sum_{m=1}^{\infty} V_{1m}\, Ms_m^{(4)}(0;\, h_1) se_m(\eta\, ; h_1^2) = -\sum_{m=1}^{\infty} V_{2m}\, Ms_m^{(4)}(0;\, h_2) se_m(\eta\, ;\, h_2^2) ,$$

and

$$4\sum i^m se_m(\beta;\, h_1^2)\, Ms_m^{(1)'}(0;\, h_1) se_m(\eta\, ;\, h_1^2) + \sum_{m=1}^{\infty} V_{1m} Ms_m^{(4)'}(0;\, h_1) se_m(\eta\, ;\, h_1^2)$$
$$= \frac{\mu_1}{\mu_2} \sum_{m=1}^{\infty} V_{2m}\, Ms_m^{(4)'}(0;\, h_2)\, se_m(\eta\, ;\, h_2^2).$$

These two equations must hold identically in η. We multiply them both by $se_n(\eta\, ;\, h_1^2)$ ($n = 1,2,3, \ldots$) and introduce the abbreviations

$$p_{nm} = \frac{2}{\pi} \int_0^{\pi} se_n(\eta\, ;\, h_1^2)\, se_m(\eta\, ;\, h_2^2)\, d\eta .$$

Making use of the orthonormalization properties

$$\int_0^{\pi} se_n(\eta\, ;\, h_1^2) se_m(\eta\, ;\, h_1^2) d\eta = \begin{cases} \frac{\pi}{2} , & n = m \\ 0 , & n \neq m \end{cases} ,$$

we obtain

$$V_{1n} Ms_n^{(4)}(0;\, h_1) = -\sum_{m=1}^{\infty} p_{nm} V_{2m}\, Ms_m^{(4)}(0;\, h_2) , \tag{13}$$

and

$$4i^n se_n(\beta;\, h_1^2) Ms_n^{(1)'}(0;\, h_1) + V_{1n} Ms_n^{(4)'}(0;\, h_1) = \frac{\mu_1}{\mu_2} \sum_{m=1}^{\infty} p_{nm}\, V_{2m} Ms_m^{(4)'}(0;\, h_2) . \tag{14}$$

Eliminating V_{1n} from (13) and (14), we have finally the infinite set of linear equations for the V_{2m}

$$4i^n se_n(\beta;\, h_1^2) Ms_n^{(1)'}(0;\, h_1) = \sum_{m=1}^{\infty} p_{nm} V_{2m} \left[\frac{\mu_1}{\mu_2} Ms_m^{(4)'}(0;h_2) + \frac{Ms_n^{(4)'}(0;h_1)}{Ms_n^{(4)}(0;h_1)} Ms_m^{(4)}(0;h_2) \right] . \tag{15}$$

From these equations the V_{2m} can be computed for given values of h_1, h_2, and β; and the V_{1m} then follow from (13). Thus the diffracted wave is completely determined.

Finally we show that our solution meets the edge condition, namely that the electromagnetic energy density must be integrable over any finite part of the space. One concludes from this that the electric and the magnetic field strenghts may become infinite at the edges $z = 0$, $x = \pm a$. In the problem in question we have the so-called symmetric case of an edge, that is, the angles around the edge, which contain the different media, are equal. Hence, as indicated by Karp [6] and shown in detail by the author in another report [7], the field strengths must behave near the edge as ρ^{t-1}, where ρ is the distance from the edge and t takes on one of the values $\frac{1}{2}$, 1, $\frac{3}{2}$,... . Expressing ξ and η near the edge by ρ and by the angle around the edge, respectively, it can be verified that in the present problem the electric field strength remains finite, since its expansion contains only powers ρ^0, $\rho^{1/2}$ etc., while the expansion of the magnetic field strength does in fact begin with a term containing $\rho^{-1/2}$.

4. Magnetic vector parallel to the slit

We now consider the other type of polarization that is, we assume that the magnetic field vector of the incident plane wave is parallel to the slit. In this case the reflected and the diffracted wave also have their magnetic field vectors parallel to the slit. If the amplitude of the incident wave is H_o, then the incident and the reflected waves H_i and H_r respectively are given by

$$\begin{aligned} \frac{H_i}{H_o} &= \exp\left[ik_1(x\cos\beta + z\sin\beta)\right] = 2\sum_{m=0}^{\infty} i^m ce_m(\beta; h_1^2) Mc_m^{(1)}(\xi; h_1) ce_m(\eta; h_1^2) \\ &\quad + 2\sum_{m=1}^{\infty} i^m se_m(\beta; h_1^2)\, Ms_m^{(1)}(\eta; h_1) se_m(\eta; h_1^2), \qquad z \geq 0, \end{aligned} \tag{16}$$

and

$$\frac{H_r}{H_o} = \exp\left[ik_1(x \cos \beta - z \sin \beta)\right] = 2\sum_{m=0}^{\infty} i^m ce_m(\beta;\, h_1^2) Mc_m^{(1)}(\xi;\, h_1) ce_m(\eta\,;h_1^2)$$

(17)

$$- 2\sum_{m=1}^{\infty} i^m se_m(\beta;\, h_1^2)\, Ms_m^{(1)}\,(\eta\,;\, h_1) se_m(\eta\,;\, h_1^2), \qquad z \geq 0 .$$

The diffracted wave, H_d, can be expanded as follows:

(18) $$\frac{H_d}{H_o} = \sum_{m=0}^{\infty} U_{1m} Mc_m^{(4)}\,(\xi;\, h_1) ce_m(\eta\,;\, h_1^2), \qquad z \geq 0 ,$$

or

(19) $$\frac{H_d}{H_o} = \sum_{m=0}^{\infty} U_{2m} Mc_m^{(4)}\,(\xi;\, h_2) ce_m(\eta\,;\, h_2^2), \qquad z \leq 0 .$$

In this expansion terms with functions $se_m(\eta\,;\, h_1^2)$ and $se_m(\eta\,;\, h_2^2)$ respectively can be omitted, as in the preceeding section. Consequently, the electric field component parallel to the plane $z = 0$ vanishes on the screen $\xi = 0$, $\eta = \pm \pi$.

The boundary conditions in the slit are now

$$(H_i + H_r + H_d)\big|_{\xi = 0, \eta > 0} = H_d\big|_{\xi = 0, \eta > 0}$$

(4b)

$$\frac{1}{\varepsilon_1} \frac{\partial}{\partial z} (H_i + H_r + H_d)\big|_{\xi = 0, \eta > 0} = \frac{1}{\varepsilon_2} \frac{\partial}{\partial z} H_d\big|_{\xi = 0,\, \eta < 0} .$$

Inserting (16), (17), (18), (19) into (4b) we obtain

$$4\sum_{m=0}^{\infty} i^m ce_m(\beta;\, h_1^2) Mc_m^{(1)}\,(\xi;\, h_1) ce_m(\eta\,;\, h_1^2) +$$

(20)

$$+ \sum_{m=0}^{\infty} U_{1m} Mc_m^{(4)}(\xi;\, h_1) ce_m(\eta\,;\, h_1^2) = \sum_{m=0}^{\infty} U_{2m} Mc_m^{(4)}(\xi;\, h_2) ce_m(\eta\,;\, h_2^2) ,$$

(21) $$\sum_{m=0}^{\infty} U_{1m} Mc_m^{(4)\prime}(\xi;\, h_1) ce_m(\eta\,;\, h_1^2) = - \frac{\varepsilon_1}{\varepsilon_2} \sum_{m=0}^{\infty} U_{2m} Mc_m^{(4)\prime}(\xi;\, h_2) ce_m(\eta\,;\, h_2^2), \quad 0 \leq \eta \leq \pi .$$

Now we multiply these equations by $ce_n(\eta\,;\, h_1^2)$ and integrate from 0 to π with respect to η. Then using the orthonormalization relations

$$\int_0^\pi ce_n(\eta\,;\,h_1^2)ce_m(\eta\,;\,h_1^2)d\eta = \begin{cases} \frac{\pi}{2} \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

and introducing the abbreviations

$$g_{nm} = \frac{2}{\pi}\int_0^\pi ce_n(\eta\,;\,h_1^2)ce_m(\eta\,;\,h_2^2)d\eta \ ,$$

we obtain the following equations:

$$(22) \qquad 4i^n ce_n(\beta;\, h_1^2)Mc_n^{(1)}(0;\, h_1) + U_{1n}Mc_n^{(4)}(0;\, h_1) = \sum_{m=0}^{\infty} g_{nm}U_{2m}Mc_m^{(4)}(0;\, h_2) \ ,$$

$$(23) \qquad U_{1n}\, Mc_n^{(4)'}(0;\, h_1) = -\frac{\varepsilon_1}{\varepsilon_2}\sum_{m=0}^{\infty} g_{nm}U_{2m}\, Mc_m^{(4)'}(0;\, h_2), \qquad n = 0, 1, 2, \dots .$$

By eliminating U_{1n}, we arrive at the infinite set of linear equations for the U_{2m}:

$$(24) \quad 4i^n ce_n(\beta;\, h_1^2)Mc_n^{(1)}(0;h_1) = \sum_{m=0}^{\infty} g_{nm}U_{2m}\left[Mc_m^{(4)}(0;h_2) + \frac{\varepsilon_1}{\varepsilon_2}\frac{Mc_n^{(4)}(0;h_1)}{Mc_n^{(4)'}(0;h_1)}\, Mc_n^{(4)'}(0;h_2)\right] ,$$

$$n = 0,1,2, \dots .$$

The edge condition of the symmetrical case is satisfied here too, as is seen by the same argument as in the previous case. The only difference is that now the magnetic field strength is finite while the electrical field strength behaves as $\rho^{-1/2}$ near the edge.

5. Numerical evaluation

We shall give here some remarks concerning the structure of the equations (15) and the procedure for their numerical solution. These remarks apply in essence also to equation (23).

When $\varepsilon_1 = \varepsilon_2$, $\mu_1 = \mu_2$, the equations are greatly simplified. In this case $p_{nm} = 1$ for $n = m$, and $p_{mn} = 0$ for $n \neq m$. The equations (15) can then be immediately solved and the diffracted wave coincides with the known result for the diffraction by a slit in homogeneous space. If $\varepsilon_1 \neq \varepsilon_2$ but $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$, that is, if $h_1 = h_2$, the same simplification arises except that the V_{2m} are equal to the V_{2m} for the homogeneous case multiplied by $\dfrac{\mu_2}{\mu_1 + \mu_2} = \dfrac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}$.

In the general case $\varepsilon_1\mu_1 \neq \varepsilon_2\mu_2$, it is convenient to introduce instead of the V_{2m} other unknowns v_m defined by

$$v_m = Ms_m^{(4)\prime}(0;\, h_2) \cdot V_{2m} .$$

Then (15) reads

$$(25) \qquad \sum_{m=1}^{\infty} p_{nm} v_m \left[\frac{\mu_1}{\mu_2} + \frac{Ms_n^{(4)\prime}(0;\, h_2)}{Ms_n^{(4)}(0;\, h_1)} \; \frac{Ms_m^{(4)}(0;\, h_2)}{Ms_n^{(4)\prime}(0;\, h_1)} \right] = 4 i^n se_n(\beta; h_1^2) Ms_n^{(1)\prime}(0;\, h_1) ,$$

$$n = 1,2,3, \dots .$$

One concludes from the definition of the p_{nm} that if one of the indices n,m is even and the other is odd then $p_{nm} \doteq 0$. So (25) separates into two independent sets, the one with $n = 1,3,5, \dots$ containing the unknowns $v_1, v_3, v_5, \dots$, the other with $n = 2,4,6, \dots$, containing the unknowns $v_2, v_4, v_6, \dots$.

A very rough estimate shows that the expression in the brackets in (25) is about of order unity, while the order of p_{nm} is less then

$$O\left(\frac{(h_1/2)^{2t}}{n^t \, t!} \right) , \quad \text{with } t = \frac{|n-m|}{2} .$$

The right side of (15) is of order $\dfrac{1}{(n-1)!} \left(\dfrac{h_1}{2}\right)^n$. Therefore the number of equations to be taken into account in the set (15) is probably of order $2h_1$. This estimate presupposes that $h_1/h_2 = O(1)$. For $h_1 \leq 5$, $h_2 \leq 5$, we suppose that one gets reasonably precise results with $n \leq 10$, that is, with two sets of five linear equations, or even fewer.

The numerical evaluation of the coefficients in (25) can be done conveniently using the 'Tables Relating to Mathieu Functions'[5]. There the parameter is $s = 4h^2$, and certain quantities $f_{o,m}$, $g_{o,m}$ are given as functions of s from which we can compute

$$\frac{Ms_n^{(4)\prime}(0;\, h)}{Ms_n^{(4)\prime}(0;\, h)} = -\, i\, (g_{o,n})^2 \cdot \left[1 - i f_{o,n}\right]$$

$$Ms_n^{(1)\prime}(0;\, h) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{g_{o,n}} \quad .$$

The $p_{n,m}$ are evaluated by using the expansion

$$se_{2t+p}(\eta\, ;\, h^2) = \sum_{r=0}^{\infty} B_{2r+p}^{2t+p}\, (h^2)\, \sin\left[(2r+p)\eta\right] , \qquad p = 0,1 \quad .$$

One gets

$$p_{2t+p,2n+p} = \sum_{r=0}^{\infty} B_{2r+p}^{2t+p}\, (h_1^2)\, B_{2r+p}^{2n+p}\, (h_2^2) , \qquad p = 0,1 \quad .$$

The coefficients B_{2r+p}^{2t+p} can be derived from the coefficients Do_{2r+p} in the above-mentioned tables;[5] we obtain

$$B_{2r+p}^{2n+p} = Do_{2r+p} \left[\sum_{r=0}^{\infty} (Do_{2r+p})^2\right]^{-1/2} \qquad p = 0,\, 1 \quad .$$

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